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Journal of Geometry and Physics 21 (1997) 169–182

JOURNAL OF GEOMETRY AND PHYSICS

Separable Hamiltonians and integrable systems of hydrodynamic type

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Received 13 July 1995

Abstract

We exhibit a surprising relationship between separable Hamiltonians and integrable, linearly degenerate systems of hydrodynamic type. This gives a new way of obtaining the general solution of the latter. Our formulae also lead to interesting canonical transformations between large classes of Stäckel systems.

Subj. Class.: Dynamical Systems
1991 MSC: 58F05, 58F07
Keywords: Separable Hamiltonians; Integrable systems; Hydrodynamics

1. Introduction

We consider a standard $2n$-dimensional symplectic manifold and a pair of functions H and F which commute with respect to the standard Poisson bracket. This means that the Hamiltonian vector fields X_H and X_F commute with respect to the standard Lie bracket. The integral curves of the Hamiltonian vector fields lie on 2-dimensional surfaces which can be co-ordinatised by the respective "times", x and t, of X_H and X_F. For non-degenerate H ($\partial^2 H/\partial p_i \partial p_j$ non-singular), we can write p_i in terms of q^i and q^i_x. We can then write the q^i components of X_F as a system of first-order PDEs in the variables q^i. When both H and F are quadratic in p_i, these equations are of hydrodynamic form. In this paper we assume

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PII S0393-0440(96)00013-7

(losing only some degenerate examples) that this system can be written in diagonal form. We show below that only linearly degenerate, semi-Hamiltonian systems can arise in this way.

We then show that each linearly degenerate, semi-Hamiltonian system can be associated with an infinite (depending upon $2n$ arbitrary functions of one variable) number of such commuting pairs of Hamiltonians. Since the latter are quadratic in momenta (and diagonalised), the corresponding Hamilton–Jacobi equation is separable. We use the Hamilton–Jacobi method to construct the *general* solution of the hydrodynamic system. Thus an arbitrary linearly degenerate semi-Hamiltonian (and not necessarily *Hamiltonian*) system of hydrodynamic type is decoupled into an infinite number of finite dimensional Hamiltonian subsystems. These Hamiltonian subsystems are just the restrictions of this hydrodynamic system to the set of stationary points of its higher integrals, which are quadratic in the first derivatives (recall that the restriction of an arbitrary evolution system to the set of stationary points of any of its higher integrals is always Hamiltonian [13,14]). The point is that an arbitrary linearly degenerate semi-Hamiltonian system of hydrodynamic type possesses infinitely many higher integrals, which are quadratic in the first derivatives. However, we prefer the finite dimensional Hamiltonian subsystems to the higher integrals, since they have a more transparent mechanical interpretation.

This is only the first paper of our programme of research, since we are left with many interesting questions. We outline two of these in the conclusions.

2. Hamiltonians with a first integral which is quadratic in momenta

We consider the standard symplectic space with canonical co-ordinates (p_i, q^i) and a pair of Poisson commuting functions $H(p,q)$ and $F(p,q)$, both of which are quadratic in p. These define a pair of symmetric quadratic forms in p, which can (generically) be simultaneously diagonalised. In this paper we assume this non-degeneracy and write H and F in diagonal form:

$$H = \frac{1}{2}\sum_{k=1}^{n} g^{kk}(q){p_k}^2 + h(q), \tag{1}$$

$$F = \frac{1}{2}\sum_{k=1}^{n} v^k(q)g^{kk}(q){p_k}^2 + f(q), \tag{2}$$

where g^{kk} is the kk-component of the inverse of the metric on some Riemannian or pseudo-Riemannian manifold. Requiring that H and F be in involution,

$$\{H, F\} = 0$$

leads directly to the following restrictions on $g^{kk}(q)$, $v^k(q)$, $h(q)$ and $f(q)$ (we assume $g^{kk} \neq 0$ and $v^i \neq v^j$ for any $i \neq j$):

$$\partial_i v^i = 0 \quad \text{for any } i = 1, \dots, n, \tag{3}$$

$$\partial_i \ln g^{kk} = \frac{\partial_i v^k}{v^i - v^k} \quad \text{for any } i \neq k, \tag{4}$$

$$\partial_i f = v^i \partial_i h \quad \text{for any } i = 1, \ldots, n, \tag{5}$$

where $\partial_i = \partial/\partial q^i$.

Now consider the equations of motion for the Hamiltonians H and F, writing x and t as their respective "times":

$$q_x^i = \frac{\partial H}{\partial p_i} = g^{ii} p_i, \quad p_{i\,x} = -\frac{\partial H}{\partial q^i}, \tag{6}$$

$$q_t^i = \frac{\partial F}{\partial p_i} = v^i g^{ii} p_i, \quad p_{i\,t} = -\frac{\partial F}{\partial q^i}. \tag{7}$$

Eqs. (6) and (7) imply the following hyperbolic system for q^i:

$$q_t^i = v^i(q) q_x^i, \quad i = 1, \ldots, n. \tag{8}$$

Thus any solution of the pair (6) and (7) generates a solution $q^i(x,t)$ of the hydrodynamic-type system (8). The variables q^i are, in fact, the Riemann invariants for system (8). It turns out that the restrictions (3)–(5) can be naturally interpreted in terms of this system. For example, the first of these means that system (8) is linearly degenerate (referred to as 'weakly non-linear' in [3]). Cross-differentiation of (4) gives

$$\partial_j \frac{\partial_i v^k}{v^i - v^k} = \partial_i \frac{\partial_j v^k}{v^j - v^k} \quad \text{for any } i \neq j \neq k \neq i. \tag{9}$$

This means, that system (8) is semi-Hamiltonian [10]. Finally, condition (5) means, that the 1-form

$$h(q)\,\mathrm{d}x + f(q)\,\mathrm{d}t$$

is an integral of system (8). Thus h is a conserved density of (8) with f the corresponding flux. We recall, that any semi-Hamiltonian system possesses infinitely many integrals of hydrodynamic type (i.e. integrals with the densities $h(q)$ independent of the derivatives q_x, q_{xx}, etc.), parametrised by n arbitrary functions of one variable (see [10] and the reviews [1,11]).

Complete integrability. The existence of H and F as above (both quadratic and simultaneously diagonalised), with v^i distinct, guarantees the existence of n (including H and F) independent integrals in involution, all of which are quadratic in momenta and in diagonal form. This is very easy to see. Suppose

$$G = \frac{1}{2} \sum_{k=1}^{n} w^k(q) g^{kk}(q) {p_k}^2 + \gamma(q)$$

is another such integral. Then w^k and γ must also satisfy (3)–(5), so that the first derivatives of w^i are given by

$$\frac{\partial_j w^i}{w^j - w^i} = \frac{\partial_j v^i}{v^j - v^i}, \quad j \neq i \quad \text{and} \quad \partial_i w^i = 0,$$

which is a system of *linear* equations for w^i (given v^i). The integrability conditions are guaranteed by conditions (3) and (4) for v^i. The n independent solutions give us n first integrals. The solution $w^i = 1$, $\forall i$, corresponds to H itself, whilst the solution $w^i = v^i$ corresponds to F. The function γ is most easily interpreted in terms of the corresponding system of hydrodynamic type:

$$q^i_\tau = w^i(q) q^i_x, \quad i = 1, \dots, n.$$

With respect to this time evolution, γ is the flux corresponding to density h. In this way we build n such integrals and time evolutions $(w^i_{(k)}, \gamma_k, \tau_k)$, $k = 1, \dots, n$, giving rise to n commuting hydrodynamic systems. We take the first of these to be $w^i_{(1)} = 1$ and $\gamma_1 = h$, corresponding to the Hamiltonian H and the simple hydrodynamic system $q^i_{\tau_1} = q^i_x$, so that $\tau_1 = x$. Similarly, $w^i_{(2)} = v^i$ and $\gamma_2 = f$, with $\tau_2 = t$, gives our system (8).

Our construction thus naturally leads us to separable Hamiltonians of Stäckel type.

Example 2.1 (The Hénon–Heiles Hamiltonian). One of the integrable cases of the Hénon–Heiles Hamiltonian is given by

$$H = \tfrac{1}{2}(P_1^2 + P_2^2) + \tfrac{1}{2} Q^1 {Q^2}^2 + {Q^1}^3. \tag{10}$$

This is the case associated with the stationary fifth-order KdV equation [5] and has a first integral which is quadratic in P:

$$F = Q^1 P_2^2 - Q^2 P_1 P_2 - \tfrac{1}{2} {Q^1}^2 {Q^2}^2 - \tfrac{1}{8} {Q^2}^4. \tag{11}$$

Whilst H generates the stationary fifth-order KdV equation (in Hénon–Heiles co-ordinates), F generates the KdV equation. The Q-parts of the corresponding Hamiltonian flows are, respectively, of the form:

$$\begin{aligned} Q^1_x &= \frac{\partial H}{\partial P_1} = P_1, & Q^2_x &= \frac{\partial H}{\partial P_2} = P_2, \\ Q^1_t &= \frac{\partial F}{\partial P_1} = -Q^2 P_2, & Q^2_t &= \frac{\partial F}{\partial P_2} = 2Q^1 P_2 - Q^2 P_1. \end{aligned}$$

Eliminating P_i, Q^1 and Q^2 satisfy the PDEs:

$$Q^1_t = -Q^2 Q^2_x, \qquad Q^2_t = -Q^2 Q^1_x + 2Q^1 Q^2_x. \tag{12}$$

Remark 2.2. Note that $Q^1_t = -1/2(Q^2)^2_x$, whilst the Hénon–Heiles equation gives $-1/2$ $(Q^2)^2 = Q^1_{xx} + 3(Q^1)^2$, so that F does indeed generate the KdV equation:

$$Q^1_t = \left(Q^1_{xx} + 3(Q^1)^2\right)_x.$$

The hydrodynamic system (12) is diagonalisable and linearly degenerate. In terms of the Riemann invariants,

$$q^1 = Q^1 - \sqrt{{Q^1}^2 + {Q^2}^2}, \qquad q^2 = Q^1 + \sqrt{{Q^1}^2 + {Q^2}^2},$$

this assumes the form

$$q_t^1 = q^2 q_x^1, \qquad q_t^2 = q^1 q_x^2.$$

The inverse of this point transformation is given by

$$Q^1 = \tfrac{1}{2}(q^1 + q^2), \qquad Q^2 = \sqrt{-q^1 q^2}.$$

In the context of our finite-dimensional Hamiltonian systems, the q^i are called parabolic co-ordinates. For a point transformation the momenta p_i are linear functions of P_1 and P_2. To preserve the symplectic form $\sum \mathrm{d}P_i \wedge \mathrm{d}Q^i = \sum \mathrm{d}p_i \wedge \mathrm{d}q^i$ this linear transformation takes the form

$$P_i = \sum_{j=1}^{2} p_j \frac{\partial q^j}{\partial Q^i}, \quad i = 1, 2,$$

and under this canonical transformation both Hamiltonians H and F become diagonal:

$$H = \frac{2q^1}{q^1 - q^2} {p_1}^2 + \frac{2q^2}{q^2 - q^1} {p_2}^2 + \tfrac{1}{8}(q^1 + q^2)({q^1}^2 + {q^2}^2), \tag{13}$$

$$F = \frac{2q^1 q^2}{q^1 - q^2} {p_1}^2 + \frac{2q^1 q^2}{q^2 - q^1} {p_2}^2 + \tfrac{1}{8} q^1 q^2((q^1 + q^2)^2 - q^1 q^2). \tag{14}$$

3. Solving the hydrodynamic system

Suppose we are given the linearly degenerate, semi-Hamiltonian system (8). We can determine the coefficients g^{kk} from Eq. (4) (which are compatible, due to (9)). Note, that g^{kk} are defined up to transformations $g^{kk} \mapsto \varphi^k(q^k) g^{kk}$, where φ^k are n arbitrary functions of one variable. Let $h(q)\,\mathrm{d}x + f(q)\,\mathrm{d}t$ be any integral of hydrodynamic type of system (8). It should be noted that semi-Hamiltonian systems have an infinite number of integrals of hydrodynamic type, (also) depending on n arbitrary functions of one variable. Choosing any of these g^{kk}, h and f, we may construct two commuting Hamiltonians H and F of the form (1) and (2). Solving the equations of motion (6) and (7), we automatically obtain a solution of the hydrodynamic-type system (8). Varying g^{kk} and the integral $h(q)\,\mathrm{d}x + f(q)\,\mathrm{d}t$, we can construct an arbitrary solution of our system. One may argue, that the solutions of hydrodynamic-type system (8) depends on only n arbitrary functions of one variable, while the arbitrariness in the construction of H and F contains $2n$ arbitrary functions (n from g^{kk} and n from the integrals). However, this gives us two different schemes of integration.

Varying the metric. Here we fix the integral (for simplicity we choose $h = f = 0$), and vary g^{kk}. Then our Hamiltonians H and F assume the form

$$H = \frac{1}{2}\sum_{k=1}^{n} g^{kk}(q)\, {p_k}^2, \qquad F = \frac{1}{2}\sum_{k=1}^{n} v^k(q) g^{kk}(q)\, {p_k}^2.$$

Note, that the equations of motion, corresponding to this Hamiltonian, H, are just the equations of geodesics for the metric $\mathrm{d}s^2 = \sum g_{kk}\, \mathrm{d}{q^k}^2$ (where $g_{kk} g^{kk} = 1$). We showed earlier that these equations admit n quadratic (in momenta) integrals and are thus separable. The Riemannian spaces corresponding to these metrics are called Stäckel spaces (see [2]).

Thus our first scheme reduces the integration of a given linearly degenerate, semi-Hamiltonian system (8) to the integration of the equations of geodesics in Stäckel spaces. Furthermore, given (8), Eqs. (3) and (4) can be solved for g^{kk}. The complete solution possesses n arbitrary functions and the geodesic equations are separable for this general metric. The solution q^i of these geodesic equations thus contains n arbitrary functions and thus constitutes the *general* solution for system (8).

Remark 3.1. The relationship between linearly degenerate, semi-Hamiltonian systems and Stäckel metrics was noted previously in [3]. Here we explain the origin of this connection.

Varying the integral. Now we take the metric g^{kk} fixed (any solution of Eq. (4)), and vary the integral. From the point of view of classical mechanics this means that we consider the motion of a particle in a Stäckel space with the metric $\mathrm{d}s^2 = \sum g_{kk}\, \mathrm{d}{q^k}^2$ fixed, but with varying potential $h(q)$. This motion has Lagrangian $L = \frac{1}{2}\sum g_{kk}(q^k_x)^2 - h(q)$ and Hamiltonian $H = \frac{1}{2}\sum g^{kk}{p_k}^2 + h(q)$. Thus our second scheme reduces the integration of the given linearly degenerate, semi-Hamiltonian system (8) to the integration of the equations of motion of a particle in a Stäckel space with varying potential. Of course, the potential $h(q)$ is not arbitrary and belongs to the class of the so-called "separable" potentials, for which the equations of motion can be integrated by the method of separation of variables. In fact, the potential h is restricted to be a conserved density of system (8), with the corresponding flux satisfying system (5). The integrability conditions lead to the system of second order, linear equations for h:

$$\partial_i\partial_j h = \frac{(\partial_j v^i)\partial_i h - (\partial_i v^j)\partial_j h}{v^j - v^i}.$$

Any solution of this gives a separable potential for the Stäckel metric g^{kk}.

3.1. An important example: $v^i = \sum_{k=1}^n q^k - q^i$

Consider the linearly degenerate, semi-Hamiltonian system:

$$q^i_t = v^i(q)q^i_x = \left(\sum_1^n q^k - q^i\right)q^i_x, \quad i = 1, \dots, n. \tag{15}$$

In this case Eq. (4) assumes the form

$$\partial_i \ln g^{kk} = \frac{1}{q^k - q^i}, \quad i \neq k,$$

leading to

$$g^{kk} = \frac{\varphi^k(q^k)}{\prod_{j\neq k}(q^k - q^j)}, \tag{16}$$

where φ^k are n arbitrary functions of one variable. The integrals $h(q)\,\mathrm{d}x + f(q)\,\mathrm{d}t$ of system (15) can also be calculated explicitly:

$$h(q) = \sum_{k=1}^{n} \frac{\psi^k(q^k)}{\prod_{j\neq k}(q^k - q^j)}, \qquad f(q) = \sum_{k=1}^{n} \frac{v^k \psi^k(q^k)}{\prod_{j\neq k}(q^k - q^j)}. \tag{17}$$

It was shown in [3] that the general solution of system (15) is given by the implicit formulae:

$$\begin{aligned}
&\int^{q^1} \frac{\xi^{n-1}\,\mathrm{d}\xi}{f^1(\xi)} + \cdots + \int^{q^n} \frac{\xi^{n-1}\,\mathrm{d}\xi}{f^n(\xi)} = x + \text{const.},\\
&\int^{q^1} \frac{\xi^{n-2}\,\mathrm{d}\xi}{f^1(\xi)} + \cdots + \int^{q^n} \frac{\xi^{n-2}\,\mathrm{d}\xi}{f^n(\xi)} = -t + \text{const.},\\
&\int^{q^1} \frac{\xi^{i}\,\mathrm{d}\xi}{f^1(\xi)} + \cdots + \int^{q^n} \frac{\xi^{i}\,\mathrm{d}\xi}{f^n(\xi)} = \text{const.}, \quad i = n-3, \ldots, 0,
\end{aligned} \tag{18}$$

where $f^i(\xi)$ are n arbitrary functions. (The constants on the right-hand sides are not essential, since the lower bounds in the integrals are not specified.)

We now demonstrate how formulae (18) can be recovered within the current framework.

The first scheme. We consider the equations of geodesics for the metric

$$\mathrm{d}s^2 = \sum g_{kk}\,\mathrm{d}q^{k^2} = \sum_{k=1}^{n} \frac{\prod_{j\neq k}(q^k - q^j)\,\mathrm{d}q^{k^2}}{\varphi^k(q^k)} \tag{19}$$

with $\varphi^k(q^k)$ arbitrary. The corresponding Hamiltonians H and F are of the form:

$$H = \frac{1}{2}\sum_{k=1}^{n} g^{kk} p_k{}^2 = \frac{1}{2}\sum_{k=1}^{n} \frac{\varphi^k(q^k) p_k{}^2}{\prod_{j\neq k}(q^k - q^j)}, \tag{20}$$

$$F = \frac{1}{2}\sum_{k=1}^{n} v^k g^{kk} p_k{}^2 = \frac{1}{2}\sum_{k=1}^{n} \frac{\left(\sum_{s=1}^{n} q^s - q^k\right)\varphi^k(q^k) p_k{}^2}{\prod_{j\neq k}(q^k - q^j)}. \tag{21}$$

According to the Hamilton–Jacobi theory, we look for a canonical transformation $(p_i, q^i) \mapsto (a_i, b^i)$ in the form $p_i = \partial S/\partial q^i$, $b^i = \partial S/\partial a_i$, where $S(q, a)$ is the generating function, satisfying the Hamilton–Jacobi equation:

$$\frac{1}{2}\sum_{k=1}^{n} \frac{\varphi^k(q^k)\left(\partial S/\partial q^k\right)^2}{\prod_{j\neq k}(q^k - q^j)} = \text{const.} \tag{22}$$

Since the metric $\mathrm{d}s^2$ is of Stäckel type, we can write S in separable form

$$S(q, a) = S_1(q^1, a) + \cdots + S_n(q^n, a),$$

after which Eq. (22) becomes

$$\frac{1}{2}\sum_{k=1}^{n}\frac{\varphi^k(q^k)\left(S_k'\right)^2}{\prod_{j\neq k}(q^k-q^j)}=\text{const.},\qquad S_k'=\frac{\partial S_k}{\partial q^k}.\tag{23}$$

It is an easy matter to show that the numerators in (23) must be the *same* function of their respective variables and that this must be polynomial:

$$\varphi^k(q^k)\left(S_k'\right)^2=r(q^k),$$

where we introduce the notation $r(\xi)=a_1+a_2\xi+\cdots+a_n\xi^{n-1}$. Hence

$$S(q,a)=\int^{q^1}\sqrt{\frac{r(\xi)}{\varphi^1(\xi)}}\,\mathrm{d}\xi+\cdots+\int^{q^n}\sqrt{\frac{r(\xi)}{\varphi^n(\xi)}}\,\mathrm{d}\xi.\tag{24}$$

In the new canonical variables a_i, $b^i=\partial S/\partial a_i$ the Hamiltonians H and F become

$$H=\tfrac{1}{2}a_n,\qquad F=-\tfrac{1}{2}a_{n-1},$$

which generate trivial flows:

$$a_{i\,x}=-\frac{\partial H}{\partial b^i}=0\ \forall i,\quad b^i_x=\frac{\partial H}{\partial a_i}=0\ \forall i\neq n,\quad b^n_x=\frac{\partial H}{\partial a_n}=\frac{1}{2},$$

and

$$a_{i\,t}=-\frac{\partial F}{\partial b^i}=0\ \forall i,\quad b^i_t=\frac{\partial F}{\partial a_i}=0\ \forall i\neq n-1,\quad b^{n-1}_t=\frac{\partial F}{\partial a_{n-1}}=-\frac{1}{2}.$$

Hence

$$\begin{aligned} b^i&=\text{const.}\quad\text{for } i=1,\ldots,n-2,\\ b^{n-1}&=-\tfrac{1}{2}t+\text{const.},\\ b^n&=\tfrac{1}{2}x+\text{const.}\end{aligned}\tag{25}$$

Differentiating (24), we find

$$b^i=\frac{\partial S}{\partial a_i}=\frac{1}{2}\int^{q^1}\frac{\xi^{i-1}}{\sqrt{r(\xi)\varphi^1(\xi)}}\,\mathrm{d}\xi+\cdots+\frac{1}{2}\int^{q^n}\frac{\xi^{i-1}}{\sqrt{r(\xi)\varphi^n(\xi)}}\,\mathrm{d}\xi.\tag{26}$$

Introducing $f^k=\sqrt{r(\xi)\varphi^k(\xi)}$ and combining (25) and (26), we arrive at formulae (18).

The second scheme. We now choose a particular metric of (16):

$$g^{kk}=\frac{R(q^k)}{\prod_{j\neq k}(q^k-q^j)},$$

where $R(\xi) = 4\prod_{j=1}^{n+1}(c^j - \xi)$, $c^j = const.$, $j = 1, \dots, n+1$. This is the metric of the unit sphere S^n:

$$x^{1^2} + \cdots + x^{n+1^2} = 1,$$

written down in the spherical–conical co-ordinates:

$$x^1 = \sqrt{\frac{\prod_{k=1}^{n}(c^1 - q^k)}{\prod_{j\neq 1}(c^1 - c^j)}}, \quad \dots, \quad x^{n+1} = \sqrt{\frac{\prod_{k=1}^{n}(c^{n+1} - q^k)}{\prod_{j\neq n+1}(c^{n+1} - c^j)}}.$$

For this choice of the metric and for the potentials (17) the corresponding Hamiltonians H and F (see (1) and (2)) assume the form

$$H = \frac{1}{2}\sum_{k=1}^{n}\frac{R(q^k){p_k}^2}{\prod_{j\neq k}(q^k - q^j)} + \sum_{k=1}^{n}\frac{\psi^k(q^k)}{\prod_{s\neq k}(q^k - q^s)}, \tag{27}$$

$$F = \frac{1}{2}\sum_{k=1}^{n}\frac{v^k R(q^k){p_k}^2}{\prod_{j\neq k}(q^k - q^j)} + \sum_{k=1}^{n}\frac{v^k\psi^k(q^k)}{\prod_{j\neq k}(q^k - q^j)}. \tag{28}$$

Hamiltonian H describes the motion of a particle on the unit sphere S^n under the action of the potential h.

Once again, we look for a canonical transformation $(p_i, q^i) \mapsto (a_i, b^i)$ in the form $p_i = \partial S/\partial q^i$, $b^i = \partial S/\partial a_i$, where the generating function $S(q, a)$ satisfies the Hamilton–Jacobi equation

$$\frac{1}{2}\sum_{k=1}^{n}\frac{R(q^k)\left(\partial S/\partial q^k\right)^2}{\prod_{j\neq k}(q^k - q^j)} + \sum_{k=1}^{n}\frac{\psi^k(q^k)}{\prod_{j\neq k}(q^k - q^j)} = \text{const.}$$

Once again, this is separable:

$$S(q, a) = S_1(q^1, a) + \cdots + S_n(q^n, a),$$

so we obtain

$$\frac{1}{2}\sum_{k=1}^{n}\frac{R(q^k)\left(S_k'\right)^2 + 2\psi^k(q^k)}{\prod_{j\neq k}(q^k - q^j)} = \text{const.}, \qquad S_k' = \frac{\partial S_k}{\partial q^k}, \tag{29}$$

with solution

$$R(q^k)\left(S_k'\right)^2 + 2\psi^k(q^k) = r(q^k),$$

where $r(\xi) = a_1 + a_2\xi + \cdots + a_n\xi^{n-1}$. Hence

$$S(q, a) = \int^{q^1}\sqrt{\frac{r(\xi) - 2\psi^1(\xi)}{R(\xi)}}\,\mathrm{d}\xi + \cdots + \int^{q^n}\sqrt{\frac{r(\xi) - 2\psi^n(\xi)}{R(\xi)}}\,\mathrm{d}\xi. \tag{30}$$

In the new canonical variables a_i, $b^i = \partial S/\partial a_i$ the Hamiltonians H and F become (as before)

$$H = \tfrac{1}{2}a_n \quad \text{and} \quad F = -\tfrac{1}{2}a_{n-1}.$$

With

$$b^i = \frac{\partial S}{\partial a_i} = \frac{1}{2}\int^{q^1} \frac{\xi^{i-1}}{\sqrt{R(\xi)(r(\xi) - 2\psi^1(\xi))}}\,\mathrm{d}\xi + \cdots + \frac{1}{2}\int^{q^n} \frac{\xi^{i-1}}{\sqrt{R(\xi)(r(\xi) - 2\psi^n(\xi))}}\,\mathrm{d}\xi,$$

we immediately arrive at formulae (18) after introducing

$$f^k = \sqrt{R(\xi)(r(\xi) - 2\psi^k(\xi))}.$$

Example 3.2 (Geodesics on a quadric). For the particular choice

$$\varphi^k(q^k) = \frac{4}{q^k}\prod_{j=1}^{n+1}(c^j - q^k), \quad c^j = \text{const.}, \quad j = 1, \ldots, n+1,$$

the metric (19) is that of the n-dimensional quadric:

$$\frac{{x^1}^2}{c^1} + \cdots + \frac{{x^{n+1}}^2}{c^{n+1}} = 1,$$

written down in the co-ordinates q^i of the lines of curvature:

$$x^1 = \sqrt{c^1\frac{\prod_{k=1}^n (c^1 - q^k)}{\prod_{j\neq 1}(c^1 - c^j)}}, \quad \ldots, \quad x^{n+1} = \sqrt{c^{n+1}\frac{\prod_{k=1}^n (c^{n+1} - q^k)}{\prod_{j\neq n+1}(c^{n+1} - c^j)}}.$$

Thus, in this particular case our first construction reduces to the famous problem of geodesics on quadrics, the integrability of which is due to Jacobi [7].

Example 3.3 (The Neumann Problem). On the other hand, for the particular choice

$$\psi^k(q^k) = c(q^k)^{n-1} - (q^k)^n, \quad c = \sum_{j=1}^n c^j,$$

the potential h assumes the form

$$h(q) = c - \sum_1^n q^k,$$

which is just the restriction of the quadratic potential $h = c^1{x^1}^2 + \cdots + c^{n+1}{x^{n+1}}^2$ from the ambient Euclidean space E^{n+1} onto the unit sphere S^n. So in this case our second construction reduces to the famous integrable problem of the motion of a particle on the unit sphere under the action of a quadratic potential, discussed by Neumann [9]. In this case Eq. (18) becomes the Jacobi inversion problem, which is known to be solvable in θ-functions.

Example 3.4 ($n = 2$). In Example 2.1 we transformed the Hénon–Heiles system (10) to the form (13) in parabolic co-ordinates. With this choice of metric, *any* choice of h and f given by (17) (with $n = 2$) can be integrated by our second scheme. In particular, if we choose $\psi^i(\xi) = \xi^n$, then

$$h_{(n)} = \frac{(q^1)^n - (q^2)^n}{q^1 - q^2} = \sum_{i=0}^{n-1} (q^1)^{n-1-i}(q^2)^i.$$

These symmetric polynomials, when written in the original (Q^1, Q^2) co-ordinates of Example 2.1 (flat co-ordinates for this metric), take the well-known form of polynomial potentials separable in parabolic co-ordinates. For instance,

$$h_{(2)} = 4(Q^1)^2 + (Q^2)^2, \qquad h_{(3)} = 2(Q^1)^3 + Q^1(Q^2)^2,$$
$$h_{(4)} = 16(Q^1)^4 + 12(Q^1)^2(Q^2)^2 + (Q^2)^4.$$

Some simple non-polynomial potentials can be obtained by putting $\psi^i(\xi) = \xi^{-n}$:

$$h_{(-n)} = -\frac{\sum_{i=0}^{n-1}(q^1)^{n-1-i}(q^2)^i}{(q^1q^2)^n}.$$

In the flat co-ordinates (Q^1, Q^2), these take the form:

$$h_{(-1)} = \frac{1}{(Q^2)^2}, \qquad h_{(-2)} = -2\frac{Q^1}{(Q^2)^4}, \qquad h_{(-3)} = \frac{4(Q^1)^2 + (Q^2)^2}{(Q^2)^6}.$$

We can, of course, choose any linear combination of these. All can be solved by using the canonical transformation (30).

4. Conclusions

We have demonstrated that any linearly degenerate semi-Hamiltonian system of hydrodynamic type

$$q^i_t = v^i(q)q^i_x$$

can be "decoupled" into an infinite number of finite-dimensional Hamiltonian subsystems, generated by a pair of commuting Hamiltonians:

$$H = \frac{1}{2}\sum_{k=1}^{n} g^{kk}p_k^2 + h, \qquad F = \frac{1}{2}\sum_{k=1}^{n} v^k g^{kk}p_k^2 + f.$$

Since the metric coefficients g^{kk} and the potentials h, f are defined up to sufficiently many arbitrary functions, it is guaranteed that any solution of the hydrodynamic system under consideration can be obtained by integrating the corresponding Hamiltonian flows

$$q_x^i = \frac{\partial H}{\partial p^i}, \quad p_x^i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad q_t^i = \frac{\partial F}{\partial p^i}, \quad p_t^i = -\frac{\partial F}{\partial q_i}$$

for appropriate Hamiltonians H and F.

Furthermore, our formulae show that *all* Stäckel systems associated with the hydrodynamic system (15) are mapped by our canonical transformation (30) (but with *arbitrary* $R(q^k)$) onto the *same* system in (a_i, b^i) co-ordinates, with solution (25). Our formulae thus provides a canonical transformation between any pair of our Stäckel systems. In particular, Examples 3.2 and 3.3 are related by canonical transformation. This is presumably different from the connection shown by Knörrer [8] (see also [12]), who showed that geodesics on a quadric can be mapped onto solutions of the Neumann problem by a Gauss map combined with an appropriate reparametrisation of trajectories. Our construction gives the more general result that (for instance) the equations of geodesics, corresponding to the Hamiltonian (20), can be mapped onto the dynamical system with the Hamiltonian (27), which describes the motion of a particle on the unit sphere under the force of a certain (in general *non-quadratic*) potential.

Eq. (15) is just one example of linearly degenerate, semi-Hamiltonian hydrodynamic system. Other examples will give rise to other Stäckel spaces, with a similar class of canonical transformations. We hope to return to these questions in a separate publication.

A similar approach applied to non-homogeneous systems,

$$q_t^i = v^i(q) q_x^i + f^i(q), \tag{31}$$

requires the Hamiltonians H and F to possess non-trivial terms which are linear in momenta:

$$H = \frac{1}{2}\sum_{k=1}^{n} g^{kk}(p_k - A_k)^2 + h, \tag{32}$$

$$F = \frac{1}{2}\sum_{k=1}^{n} v^k g^{kk}(p_k - A_k)^2 + \sum_{k=1}^{n} f^k p_k + f. \tag{33}$$

The corresponding Hamiltonian flows

$$q_x^i = \frac{\partial H}{\partial p_i} = g^{ii}(p_i - A_i) \quad \text{and} \quad q_t^i = \frac{\partial F}{\partial p_i} = v^i g^{ii}(p_i - A_i) + f^i$$

immediately imply

$$q_t^i = v^i q_x^i + f^i.$$

We emphasise, that this approach is applicable only if the "homogeneous" part of system (31) is linearly degenerate and semi-Hamiltonian. It turns out, however, that if $f^i(q)$ are non-zero, then there exist at most finitely many (up to canonical transformations $\mathbf{q} \mapsto \mathbf{q}, \mathbf{p} \mapsto \mathbf{p} + \operatorname{grad}\sigma(q)$) pairs of commuting Hamiltonians (32) and (33). However, this gives a straightforward procedure of constructing a multiparameter family of particularly interesting solutions of the system under consideration.

Example 4.1 (Gibbons–Tsarev system). Gibbons and Tsarev [6] recently considered the finite reductions of the Benney moment equations. They found that the 2-reductions are given in terms of the solutions of the non-homogeneous hydrodynamic system:

$$q_t^1 = q^2 q_x^1 + \frac{1}{q^1 - q^2}, \qquad q_t^2 = q^1 q_x^2 + \frac{1}{q^2 - q^1}. \tag{34}$$

One can show, that the only possibility for H and F is the following:

$$H = \frac{p_1^2 - p_2^2}{q^1 - q^2} + \delta(q^1 + q^2),$$

$$F = \frac{q^2 p_1^2 - q^1 p_2^2}{q^1 - q^2} + \frac{p_1 - p_2}{q^1 - q^2} + \delta q^1 q^2, \qquad \delta = \text{const.}$$

Thus H and F are defined uniquely up to an arbitrary constant δ.

This example generalises to the n-component case:

$$q_t^i = v^i q_x^i + \frac{1}{\prod_{k\neq i}(q^i - q^k)} \quad \text{where } v^i = \sum_1^n q^k - q^i.$$

Here H and F are of the form

$$H = \sum_{i=1}^n \frac{p_i^2}{\prod_{k\neq i}(q^i - q^k)} + \delta \sum_{i=1}^n q^i,$$

$$F = \sum_{i=1}^n \frac{v^i p_i^2}{\prod_{k\neq i}(q^i - q^k)} + \sum_{i=1}^n \frac{p_i}{\prod_{k\neq i}(q^i - q^k)} + \delta \sum_{i,k=1}^n (1 - \delta_{ik}) q^i q^k,$$

where $\delta = \text{const.}$

We discuss non-homogeneous systems in more detail in [4], where we show that the functions f^i are just the components of a Killing vector of the metric g_{ii}.

Acknowledgements

We thank the Royal Society for their financial support of APF to Moscow and EVF to Leeds, making this collaboration possible.

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